PAPER

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PAPER

Dynamics of a charged Thomas oscillator in an external magnetic field

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Abstract

In this letter, we provide a detailed numerical examination of the dynamics of a charged Thomas oscillator in an external magnetic field. We do so by adopting and then modifying the cyclically symmetric Thomas oscillator to study the dynamics of a charged particle in an external magnetic field. These dynamical behaviours for weak and strong field strength parameters fall under two categories; conservative and dissipative. The system shows a complex quasi-periodic attractor whose topology depends on initial conditions for high field strengths in the conservative regime. There is a transition from adiabatic motion to chaos on decreasing the field strength parameter. In the dissipative regime, the system is chaotic for weak field strength and weak damping but shows a limit cycle for high field strengths. Such behaviour is due to an additional negative feedback loop that comes into action at high field strengths and forces the system dynamics to be stable in periodic oscillations. For weak damping and weak field strength, the system dynamics mimic Brownian motion via chaotic walks. We claim that the modified Thomas oscillator is a prototypical model to understand the dynamics of an active particle.

1. Introduction

Thomas system is an interesting chaotic system with cyclic symmetry in its state variables and a single parameter that stands for frictional damping. Physically the model represents a point particle in a force field under the action of some source of energy. The importance of the system is that one can understand the origin of deterministic chaos in terms of feedback circuits[1, 2]. The cyclically symmetric Thomas system is given by equation (1)

$$\begin{aligned}\frac{dX}{dT} &= -bX + \sin(Y)\\ \frac{dY}{dT} &= -bY + \sin(Z)\\ \frac{dZ}{dT} &= -bZ + \sin(X)\end{aligned} \tag{1}$$

where b is the damping parameter. The route to chaos and symbolic dynamics are discussed in[3].

Whenever a variable influences its own rate of increase or decay directly or via other system variables, there exists a feedback circuit. The feedback circuits can be readily read off from the Jacobian matrix of the system. In general, a non-zero entry a_{ij} implies that the variable x_j exerts an influence on variable x_i. This influence can be negative or positive depending on the sign of the entry and one can talk about positive or negative feedback circuits. If the non-zero entries of the Jacobian are such that their row(i) and column(j) indices form cyclic permutations of each other they are said to form a feedback circuit. The feedback circuit can be positive or negative depending on the parity of the number of negative interactions involved in the loop. Positive circuits have even negative interactions, whereas negative feedback circuits have odd numbers of negative interactions.

In contrast with the linear system, where the entries are constant, for a non-linear system, at least one term in the Jacobian will be a function of one or more variables and therefore the nature of feedback loops depends on location in phase space. It has been proved that a positive feedback circuit is essential for generating multi-stationarity and at least one negative circuit of two or more elements is required for stable periodic oscillations [2, 4, 5]. There can be more than one feedback circuit in the system, and the operation of one circuit can be hampered by the presence of other circuits in the system when one or more parameters of the system are changed[1].

The Jacobian for Thomas system is given by

$$D_g = \begin{bmatrix} -b & \cos(Y) & 0 \\ 0 & -b & \cos(Z) \\ \cos(X) & 0 & -b \end{bmatrix} \tag{2}$$

where, $\mathbf{g}$ is the Thomas system given in equation (1). We see that each variable influences its own evolution, decreasing its own rate, and there is a three-element feedback circuit($a_{12} \rightarrow a_{23} \rightarrow a_{31}$). This single three-element feedback circuit (positive or negative) is sufficient to produce chaos(a minimal requirement) in this system depending on its location in phase space[2]. The $b = 0$ case, the conservative limit, is an example of three-dimensional fractional Brownian motion in a purely deterministic system and it is the only example of this sort where fractional Brownian motion is connected to non-linear feedback circuits[3, 6]. The properties, as mentioned earlier, of the system make it suitable for applications in chemistry as representative autocatalytic reactions[7], ecology[8], and in evolution[9]. Spatio-temporal patterns are observed for many such oscillators with a non-linear coupling scheme in[10] while with a linear coupling and non-identical oscillators in [6].

Exceptional properties materialise when a charged particle interacts with an external magnetic field. The use of an external magnetic field enables us to constrain and order the particle's motion. Whether it is a study related to astrophysical plasma or the Brownian motion of charged particles in a fluid in the laboratory, the most straightforward situation will be to understand a single particle dynamics in a given external field. This will allow us to understand many such particles' motion in cohesive groups. This paper considers a study on the dynamics of a charged Thomas oscillator in an external magnetic field with the assumption that the state variables are components of velocities. With the magnetic field switched on, we have one more control parameter to control the dynamics and chaos in the system. We study the dynamics by considering two cases, conservative($b = 0$) and dissipative($b \neq 0$), for weak and strong field strength parameters. Unidirectional and isotropic constant magnetic fields have been used in this computational study.

The paper is organized in the following way. In (2), we discuss the modelling of a charged Thomas oscillator. Section (3) comprises an elaborate discussion on the dynamics of the oscillator for zero damping with unidirectional and isotropic magnetic fields. In (4), We consider non-zero damping and study the dynamics for both unidirectional and isotropic magnetic fields. Section (5) is for our various results and discussion.

2. Modelling and dimensional analysis for a charged Thomas oscillator in an external magnetic field

Consider the motion of a charged particle in a fluid explicated by equations of motion given below, assuming a constant magnetic field along the z-direction[$B_a = B_0\hat{k}$]. Here, m is the mass of the particle, γ, the damping coefficient, F, is the strength of the interaction force field and the v_i's, where $i = x, y, z$, are the components of velocities. The components of velocities are coupled with a cyclic symmetry.

$$m\frac{dv_x}{dt} = -\gamma v_x + F\sin\left(\frac{v_y}{v_c}\right) \pm |q|v_y B_0 \tag{3}$$

$$m\frac{dv_y}{dt} = -\gamma v_y + F\sin\left(\frac{v_z}{v_c}\right) \mp |q|v_x B_0 \tag{4}$$

$$m\frac{dv_x}{dt} = -\gamma v_z + F\sin\left(\frac{v_x}{v_c}\right) \tag{5}$$

where B_0 is the magnitude of the applied magnetic field along the *z-direction*. The upper sign is for the positive charge and the lower sign is for a negative charge. Define dimensionless velocities and time as

$$X = \frac{v_x}{v_c},\ Y = \frac{v_y}{v_c},\ Z = \frac{v_z}{v_c},\ T = \frac{t}{\tau} \tag{6}$$

where, v_c is the characteristic speed and τ is the characteristic time such that the characteristic length scale can be defined to be $l = v_c\tau$. Putting equation (6) in equation (3), one will get the following.

$$\frac{dX}{dT} = -\frac{\gamma v_c \tau}{m v_c} X + \frac{F\tau}{m v_c}\sin(\mathrm{Y}) \pm \frac{|\mathrm{q}|\mathrm{B}_0 \mathrm{v}_c \tau}{\mathrm{m}\mathrm{v}_c}\mathrm{Y}$$
$$= -\frac{\gamma\tau}{m} X + \frac{F\tau}{m v_c}\sin(\mathrm{Y}) \pm \frac{|\mathrm{q}|\mathrm{B}_0 \tau}{\mathrm{m}}\mathrm{Y} \tag{7}$$

The dimensionless parameters are defined to be $b = \frac{\gamma\tau}{m}$(the damping parameter), $c = \frac{|q|B_0\tau}{m}$(field strength parameter) and set $\frac{F\tau}{m v_c} = 1$. Following the similar procedure for equations (4)–(5) one will end up with a modified Thomas oscillator which is charged and placed in an external magnetic field along the z-direction.

$$\frac{dX}{dT} = -bX + \sin(Y) \pm cY$$
$$\frac{dY}{dT} = -bY + \sin(Z) \mp cX$$
$$\frac{dZ}{dT} = -bZ + \sin(X) \tag{8}$$

For the Isotropic magnetic field[$B_a = B_0(\hat{i} + \hat{j} + \hat{k})$] we have

$$\frac{dX}{dT} = -bX + \sin(Y) \pm c(Y - Z)$$
$$\frac{dY}{dT} = -bY + \sin(Z) \pm c(Z - X)$$
$$\frac{dZ}{dT} = -bZ + \sin(X) \pm c(X - Y) \tag{9}$$

In a compact form, one can write,

$$f(x; b, c) = g(x; b) + h(x; c) \tag{10}$$

where **x** stands for the dimensionless velocities and b & c are dimensionless parameters as defined above. **g**(**x**;b) is the Thomas system and **h**(**x**;c) is the Lorentz force term in the dimensionless form.

For a positive charge, the Jacobian matrix corresponding to equations (8) and (9) are respectively

$$Df_{uni} = \begin{bmatrix} -b & \cos(Y) + c & 0 \\ -c & -b & \cos(Z) \\ \cos(X) & 0 & -b \end{bmatrix} \tag{11}$$

$$Df_{iso} = \begin{bmatrix} -b & \cos(Y) + c & -c \\ -c & -b & \cos(Z) + c \\ \cos(X) + c & -c & -b \end{bmatrix} \tag{12}$$

3. Dynamics for zero damping (b = 0)

3.1. Unidirectional magnetic field

This section considers the effect of a unidirectional magnetic field on a charged Thomas oscillator under zero damping. The applied magnetic field direction is chosen to be along the z-direction, $B_a = B_0\hat{k}$. The dynamical equations, in this case, will be given by setting $b = 0$ in equation (8). The flow, generated by this system of coupled equations, is conservative since the divergence of flow is zero, $\nabla \cdot f = 0$. Further, the sum of the Lyapunov exponents, in this case, is zero supporting this fact. With $b = 0$ we have only one control parameter here, the field strength parameter c, by tuning which one can explore the complex dynamics of the system. From figure 1(a), it is observed that the system behaviour is chaotic for $c < 2$, and the largest Lyapunov exponent(blue) is positive in this range. After that, all the exponents are zero, pointing towards complex quasi-periodic oscillations($c > 2$) close to regular dynamics figure 1(b). The system's phase space for $c = 5$ is given in figure 2.

Since the system is conservative, the mechanism leading towards chaos is different from dissipative systems. In conservative systems, the phase volume must be conserved, leading towards energy conservation, so the trajectory should fold back into the phase space. An important related phenomenon is the phase-locking onto non-linear resonance. In this phenomenon, if the ratio of two frequencies associated with the dynamics approaches an integer or otherwise a rational fraction non-linearities cause phase locking of nearby trajectories onto this resonance[11].

According to classical electromagnetic theory, a particle, with charge q, mass m and velocity v, in an external magnetic field, experiences Lorentz force which acts perpendicular to both the external magnetic field vector and the particle's velocity vector. In a strong uniform magnetic field B', the motion of the charged particle is

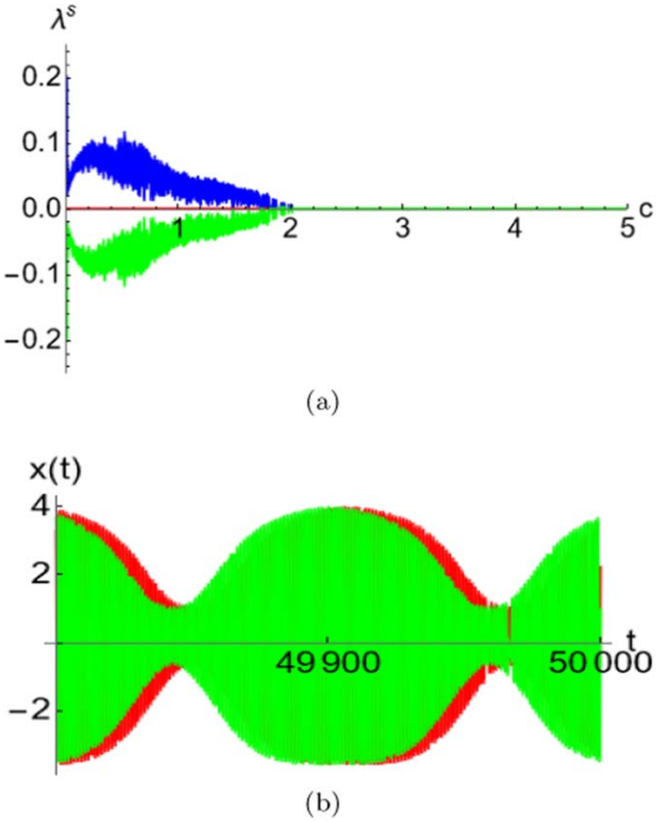

Figure 1. (a) Lyapunov spectrum(Largest Lyapunov exponent(blue)) (b) Time series showing linearly diverging trajectories for two nearby initial conditions showing complex quasi-periodic oscillations for $c = 5$.

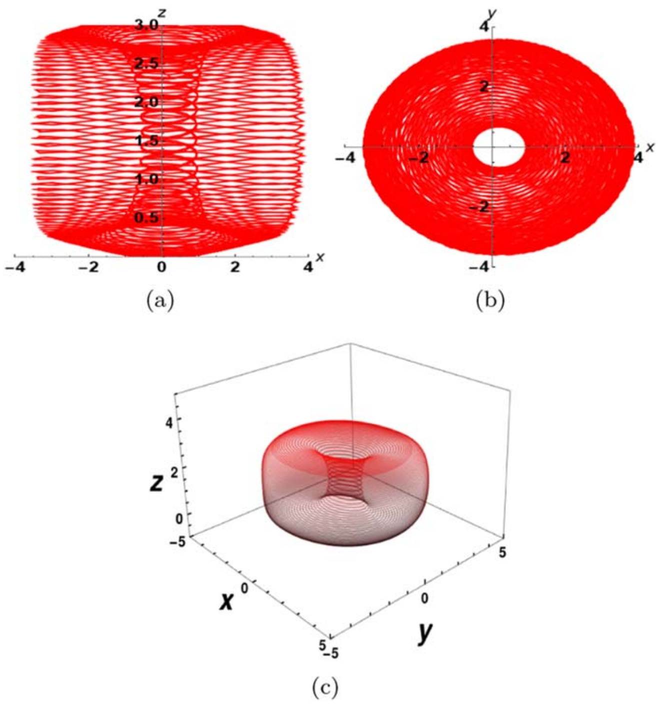

Figure 2. Phase space plots for a charged Thomas oscillator under a unidirectional magnetic field with $c = 5$ (a)X-Z phase plot (b) X-Y phase plot (c) The complex quasi-periodic attractor.

such that they gyrate around the direction of the external magnetic field. The frequency of the gyrated motion will be given by

$$\omega_c = \frac{qB'}{m} = \frac{2\pi r_c}{\mathcal{T}} \tag{13}$$

and is also known as the cyclotron frequency, where r_c is the cyclotron radius and $\mathcal{T}$ is the period of cyclotron motion. The frequency of gyration depends on the strength of the magnetic field B', charge and mass of the particle. In the absence of other external forces, in a homogeneous magnetic field, the motion of the charged particle will be a combination of gyration as well as uniform rectilinear motion, which will result in a helical trajectory with a given frequency along the field lines. The radius of gyration will be smaller wherever the field strength is a maximum and larger wherever the field strength is a minimum[12].

A moving lump of charge is like a current flow that can generate a magnetic field around it. We call this an induced magnetic field B_i. The strength of the induced magnetic field is decided by the initial velocities, acceleration of the charge, amount of charge, and the strength of the external magnetic field. So for a fixed charge and mass, the only way one can control the induced magnetic field is by choice of initial conditions and by varying the external field. Due to the superposition of the two magnetic fields, the resultant magnetic field $B = B_a + B_i$ is inhomogeneous as shown in figure 10. There is a coaxial double-helical motion, with the inner helix diverging out into the outer helix and vice-versa. The convergence and divergence happen at opposite poles due to drift motion, and they can switch polarity depending on the nature of the charge and/or the direction of the magnetic field. Also, because this drift motion is accelerated, radiations will be emitted at both ends of these helixes. One can say the particle is trapped in an external magnetic field.

Whenever the particle moves through an inhomogeneous magnetic field, a force will act at the right angle to the field, giving the particle a direction of motion perpendicular to both the force and the magnetic field. The resultant inhomogeneous magnetic field can be parallel or perpendicular to the direction of the applied magnetic field. Perpendicular inhomogeneity means there are compressions and rarefactions of field lines. The variations in the field parallel to the applied field direction lead to curved field lines. The former leads to gradient drift and the latter to centrifugal drift. Assume $\mathbf{R}$ is the radius of curvature of the magnetic field lines, then the magnitudes and directions of these two drifts are given by

$$v_G = \frac{-mV_\perp^2}{2qB^2}(\mathbf{B} \times \nabla B) \tag{14}$$

$$v_C = \frac{-mV_\parallel^2}{qB^2R^2}(\mathbf{R} \times B) \tag{15}$$

The minus sign here indicates that the particle's motion will be opposite for positive and negative charges. One can distinguish between the drift motion and cyclotron motion in the presence of a strong applied magnetic field. In the presence of a strong magnetic field, the distortion of cyclotron orbits due to inhomogeneities or other forces will be small. The changes in period and the radius of gyration are then small compared to the characteristic time scale and length scales that characterize all other relevant quantities that characterize these orbits. The motion that satisfies all these conditions is called adiabatic motion ie, $\omega_c \gg \frac{1}{\tau}$, $r_c \ll l$.

When the particle's motion satisfies the adiabatic conditions, one can treat it as a combination of three mutually exclusive motions: the rectilinear motion along the field, the gyrated motion around the field, and the drift motion in a direction perpendicular to the field. Suppose the condition for adiabatic motion is not satisfied then there will be confusion among the latter two motions leading to complex dynamics and chaos. In this case, for the dynamics of the charged Thomas oscillator, the adiabatic conditions are not satisfied in the limit of the weak applied magnetic field. As we decrease the magnetic field strength parameter, the system falls onto non-linear resonance and from there to chaos. In this system, when the strength of the field strength parameter is decreased, the strength of the non-linear resonance will increase, and they will overlap. As a result, it will never follow a smooth regular path in the phase space but wander chaotically. The smooth curves on the invariant-tori break because of the competition between these non-linear resonances. Overlap of non-linear resonance leads to the onset of chaos. On decreasing the field strength chaos appears in the neighbourhood of non-linear resonance.

Projections of the chaotic attractors for different values of c are given in figure 3. The corresponding bifurcation of the system with the field strength parameter is shown via the Poincaré sections. As we see from figure 4(a), for the weak magnetic field strength parameter, we have a chaotic sea. We can see mixing for intermediate values of the parameter, and for higher values, the trajectories are nearing regular trajectories. From figures 4(b), (c), one can observe quasi-periodic orbits surrounded by a KAM surface. From the Lyapunov spectra in figure 1, the Lyapunov exponents are zero for higher values of the parameter characterising complex quasi-periodic oscillations, very near to regular motion. The resonance overlap criteria make use of the fact that when distinct resonances overlap, no KAM tori can exist between them. The strength of the external field at

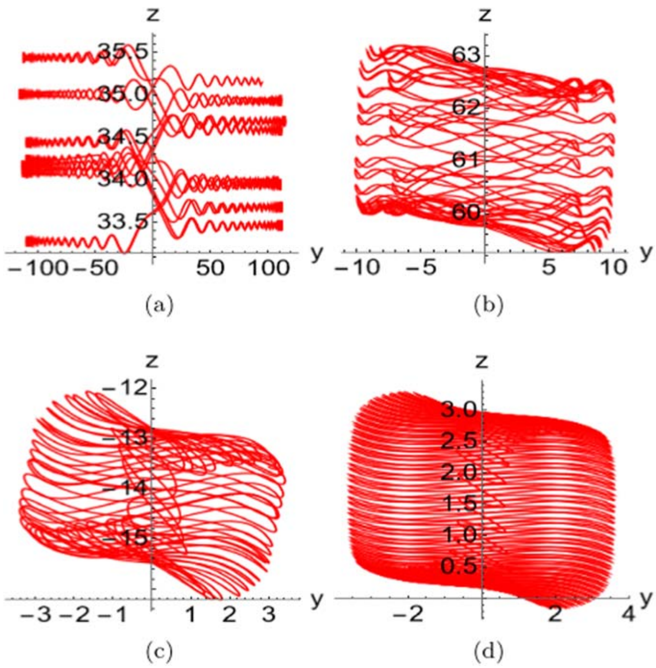

Figure 3. The figure shows the projections of attractors on to $y - z$ plane for magnetic field strength parameter (a) c = 0.3(Chaotic Motion) (b) c = 1.2(Chaotic Motion) (c) c = 1.3(Chaotic Motion) (d) c = 3.5 (Complex Quasi-periodic Motion)respectively for constant unidirectional external magnetic field .

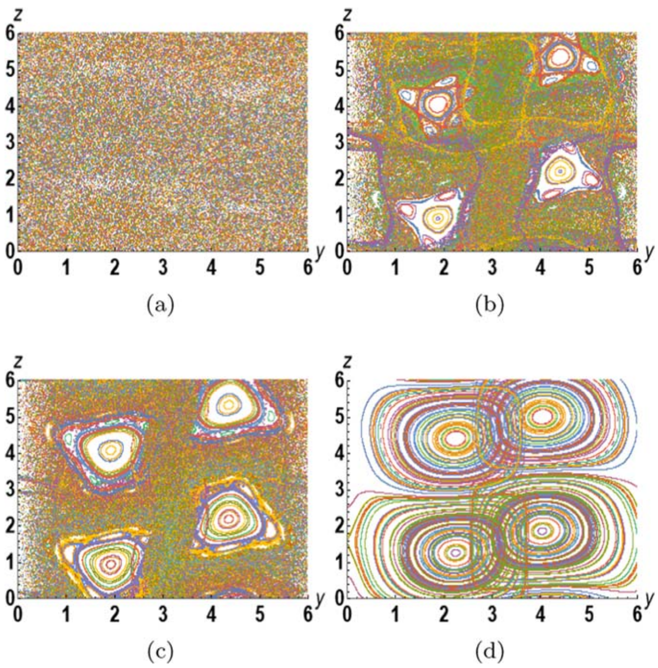

Figure 4. The figure shows the bifurcation taking place in the system via Poincaré—sections for magnetic field strength parameter (a) c = 0.3(Chaotic Motion) (b) c = 1.2(Chaotic Motion) (c) c = 1.3(Chaotic Motion) (d) c = 3.5(Complex Quasi-periodic Motion) respectively for constant unidirectional external magnetic field .

which this occurs marks the commencement of chaos. The power spectrum of the system for the same parameter values is also given in figure 5. A wide frequency distribution portrays the chaotic nature of the system, whereas, for regular systems, the power spectrum consists of single lines at harmonics and sub-harmonics.

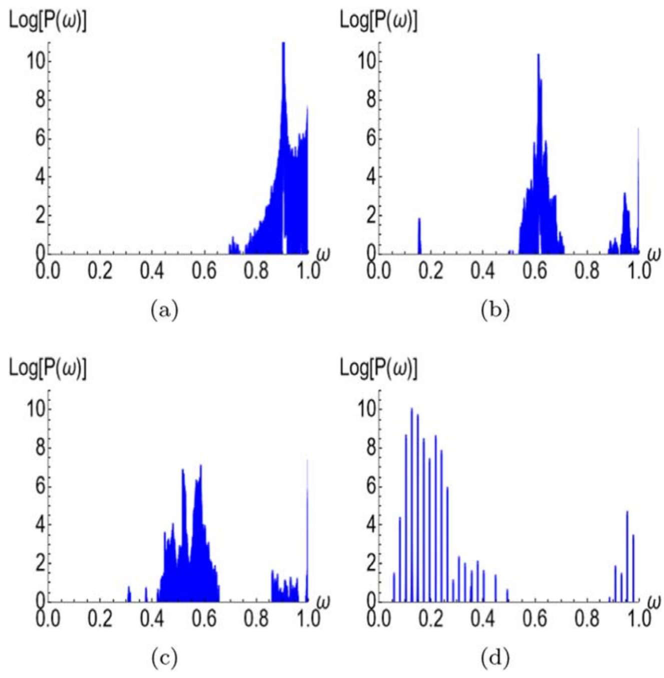

Figure 5. The figure shows the power spectra for magnetic field strength parameter (a) c = 0.3(Chaotic Motion) (b) c = 1.2(Chaotic Motion) (c) c = 1.3(Chaotic Motion) (d) c = 3.5(Complex Quasi-periodic Motion) respectively for constant unidirectional external magnetic field .

3.2. Isotropic magnetic field

The dynamics for an isotropic external magnetic field are given by setting $b = 0$ in equation (9). This case also shows conservative dynamics. At higher values of the field strength parameter, the system reaches the complex quasi-periodic attractor much earlier than that in the previous case, $c > 0.97$, below which it is chaotic. Most importantly, the attractor gets aligned with the resultant field direction. The Lyapunov spectrum is given below in figure 6.

The projections of attractors are shown in figure 7 and the bifurcation diagram via Poincaré sections is in figure 8. The corresponding power spectrum is given in figure 9.

3.3. The effect of the feedback circuit & Lorentz forcing in the conservative case

Because the shape of the attractor is not sensitive to the exact nature of non-linearities, one can use a Taylor series expansion to write down an approximate version of equation (8) for the conservative case. This equation works very similarly to the original equation.

$$\begin{aligned}\frac{dX}{dT} &\approx Y - Y^3 + cY\\ \frac{dY}{dT} &\approx Z - Z^3 - cX\\ \frac{dZ}{dT} &\approx X - X^3\end{aligned} \tag{16}$$

The corresponding Jacobian Matrix is given by

$$\mathbf{Df_{uni}} \approx \begin{bmatrix} 0 & 1 - 3Y^2 + c & 0\\ -c & 0 & 1 - 3Z^2\\ 1 - 3X^2 & 0 & 0 \end{bmatrix} \tag{17}$$

Each of the entries of type $1 - 3x^2$ is positive for $|x| < \frac{1}{\sqrt{3}}$ and negative for $|x| > \frac{1}{\sqrt{3}}$. There are two feedback circuits, a three-element feedback circuit formed by the product of matrix entries $a_{12} = 1 - 3Y^2 + c$, $a_{23} = 1 - 3Z^2$, $a_{31} = 1 - 3X^2$ and a two-element negative feedback circuit formed by the product of matrix entries $a_{12} = 1 - 3Y^2 + c$, $a_{21} = -c$. As far as the three-element feedback circuit is concerned, it can be positive or negative depending on the values of state variables, i.e., the location in phase space. It can generate chaotic behaviour for a suitable choice of parameter c. At $c = 0$, the system has infinitely many unstable steady states, and the trajectories wander around the phase space. In the limit $t \longrightarrow \infty$, the trajectories fill the entire phase space via chaotic walks known as Labyrinth chaos. It is confirmed by the only positive Lyapunov exponent($\lambda_1 \approx 0.091$)

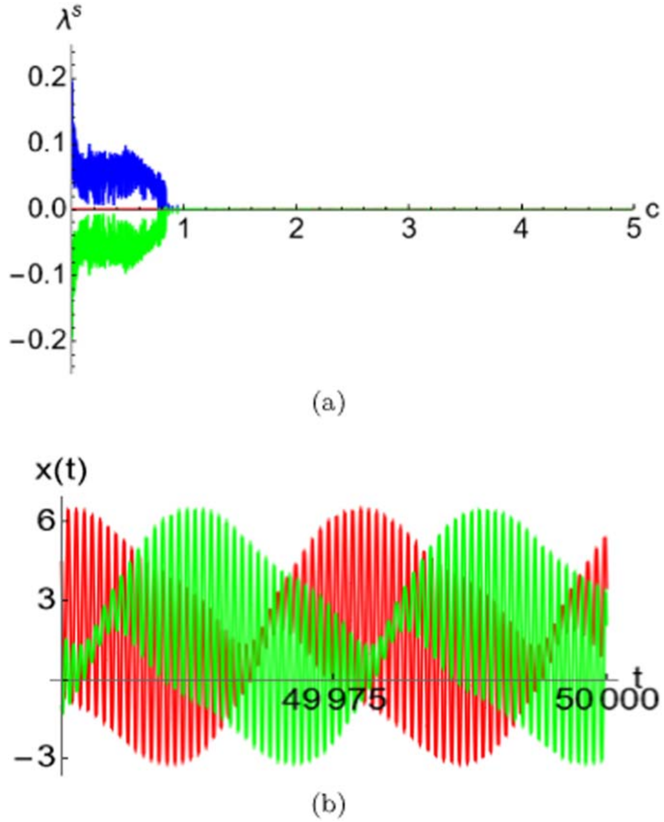

Figure 6. (a) Lyapunov spectrum (Largest Lyapunov exponent(blue))(b) Time series showing linearly diverging trajectories for two nearby initial conditions showing complex quasi-periodic oscillations for $c = 5$.

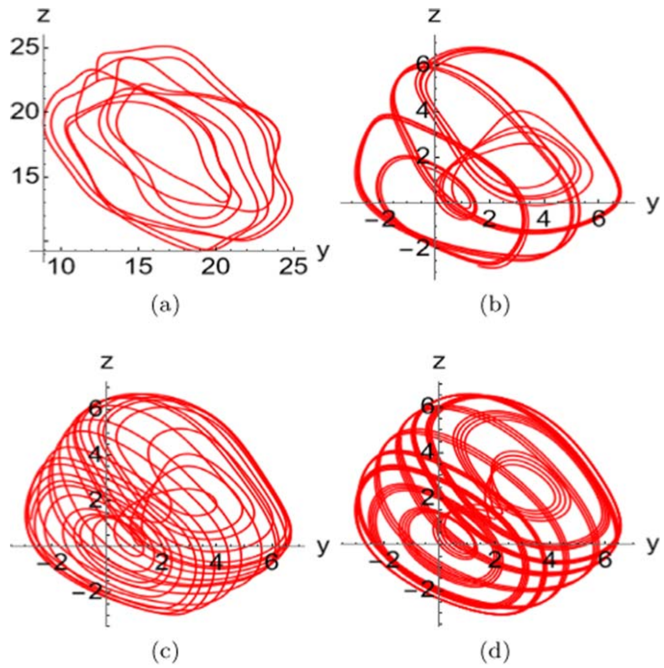

Figure 7. The figure shows the projections of attractors on to $y - z$ plane for magnetic field strength parameter (a) c = 0.3(Chaotic Motion) (b) c = 0.7(Chaotic Motion) (c) c = 0.9(Complex Quasi-periodic Motion) (d) c = 1.5(Complex Quasi-periodic Motion) respectively for constant isotropic external magnetic field.

at $c = 0$. As we increase the value of c(increase Lorentz forcing) smoothly from zero, the size of the attractor steadily decreases. Finally, around $c \approx 2$, the trajectories start to move back and forth between two steady states located at the origin (0, 0, 0) and (0, 0, π). One can consider this as an equilibrium which becomes much more

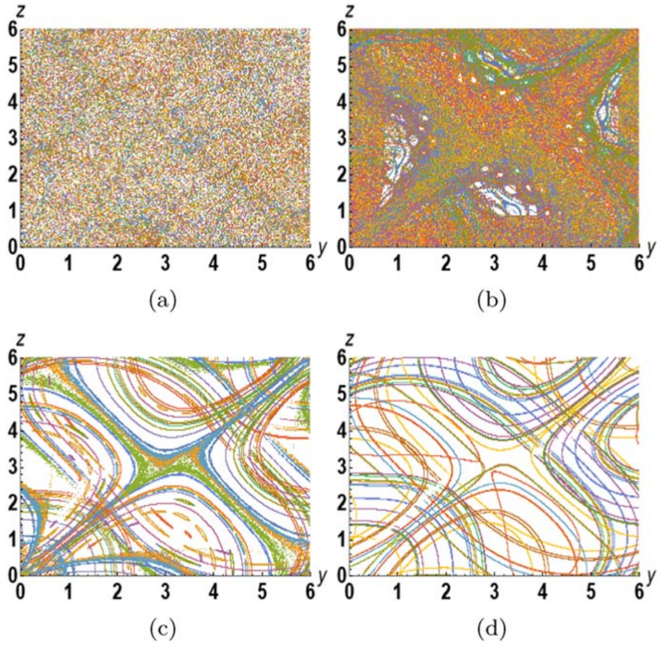

Figure 8. The figure shows the bifurcation taking place in the system via Poincaré—sections for magnetic field strength parameter (a) c = 0.3(Chaotic Motion) (b) c = 0.7(Chaotic Motion) (c) c = 0.9(Complex Quasi-periodic Motion) (d) c = 1.5(Complex Quasi-periodic Motion) respectively for constant isotropic external magnetic field.

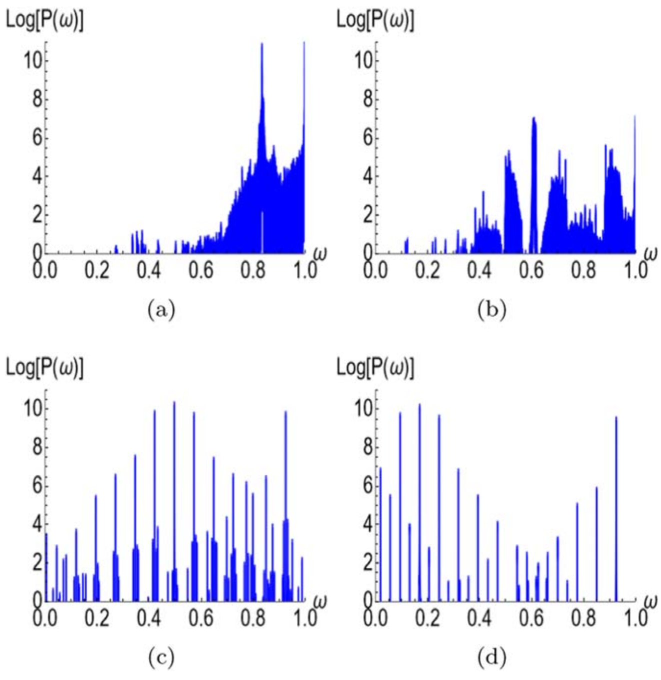

Figure 9. The figure shows the power spectra for magnetic field strength parameter (a) c = 0.3(Chaotic Motion) (b) c = 0.7(Chaotic Motion) (c) c = 0.9(Complex Quasi-periodic Motion) (d) c = 1.5(Complex Quasi-periodic Motion) respectively for constant isotropic external magnetic field.

pronounced at very high field strength. Eigenvalue calculation shows that the former is a saddle focus of the type $(+/--)$, and the latter is of type $(-/++)$. More specifically, $(+/--)$ means that there is a positive real root and a pair of complex roots whose real values are negative. On the other hand, $(-/++)$ means there is a negative real root and a pair of complex roots with negative real parts. Therefore, the dynamical behaviour near the steady state $(0, 0, 0)$ is such that it is periodically attractive in the *XY*-plane but $(0, 0, \pi)$ is periodically repulsive. The two-element feedback circuit in the vicinity of these steady states behaves in the following way. A trajectory that moves near the steady state $(0, 0, 0)$ will spiral inward in the *XY*-plane but repelled along the *z*-

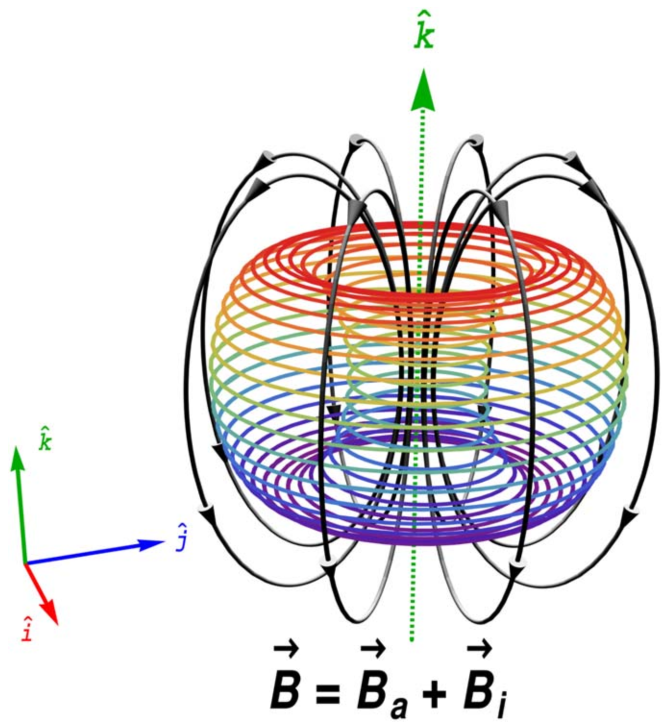

Figure 10. Artistic Representation of the Attractor with Magnetic Field Lines. $\hat{k}$ is a unit vector in the direction of the applied magnetic field.

direction due to its inherent nature. On the other hand, the trajectories nearing $(0, 0, \pi)$ will spiral out first and then repelled along the z-direction, and the cumulative result is a quasi-periodic trajectory. The unstable periodicity in the XY-plane is related to the linear nature of the Jacobian matrix at high values of c. The eigenvalues evaluated in the vicinity of these steady states did not show any change of sign in the real part of the complex eigenvalues, and the non-linear effects subsided entirely. This leads only to spiralling-in/spiralling-out without settling into a closed orbit and hence a quasi-periodic trajectory.

The role of negative feedback can be compared with a thermostat. A thermostat maintains the temperature of the system near a desired fixed value. By self-regulating, the system maintains equilibrium. Similarly, the negative feedback self-regulates specific properties, say, the energy in the modified Thomas system. According to Charles A. Coombes's conclusion drawn in [13], a charged particle receives energy from the source of a magnetic field if it is time-varying. Figure 10 shows that the magnetic field (the background field in black lines) is not uniform. That means if a charged particle moves in this field, the background changes w.r.t time as the particle moves along the direction of the field. Hence, the particle can exchange energy depending on the field strength. Here, we can also see that the magnetic flux (i.e. lines of force passing through a unit area) is higher in the central region. Therefore, when a point charge moves upward in the direction of the filed lines away from the centre, it continuously loses energy and, gradually, falls into bigger radii of the inner helix until its z-component of velocity becomes zero. This happens because as it moves upward, it experiences that the background magnetic field is decreasing due to the diverging field lines. As a result, the point charge will gain less energy(or dissipate more energy than the gain) from the source compared to the central region. Hence, the helicity increases until the direction of the background field flips. Now, the particle moves in the outer magnetic field in a downward direction. However, unlike in the former case, field lines diverge most in the middle in the outer region. Hence we get the maximum helicity for the particle in the middle. Then as it moves further below, it experiences an increase in field strength due to a gradual increase in the magnetic flux. The particle will then gain energy from the source and moves into shorter radii gradually until the field flips again. Then it ends up in the inner helix repeating the same behaviour.

For the conservative isotropic case, one can follow the above arguments to describe the observed dynamics. the nature of the steady states and the feedback circuit kept in the vicinity of these steady states decide the nature of trajectories. From an approximated version of the Jacobian matrix given in equation (12) with $b = 0$, one can figure out that there are two three-element feedback circuits(equation (18)).

$$\mathbf{Df}_{\mathbf{iso}} \approx \begin{bmatrix} 0 & 1 - 3Y^2 + c & -c \\ -c & 0 & 1 - 3Z^2 + c \\ 1 - 3X^2 + c & -c & 0 \end{bmatrix} \tag{18}$$

For small values of c, the feedback circuit formed by the product of matrix entries ($a_{12} = 1 - 3Y^2 + c$, $a_{23} = 1 - 3Y^2 + c$, $a_{31} = 1 - 3Y^2 + c$), which can be positive or negative depending on the value of state variables, dominates over the constant negative feedback circuit formed by the product of the other matrix entries ($a_{13} = -c$, $a_{21} = -c$, $a_{32} = -c$). This generates chaotic dynamics for a wide range of field strength parameters $c < 0.97$[The negative three-element feedback circuit mentioned above cannot generate chaos. But it can play a role in deciding the dynamic nature when the parameter of the system is changed]. When $c > 0.9$ the trajectories starts to move back and forth between two steady states located at (0, 0, 0) and (π, π, π). The eigenvalue calculation of the Jacobian shows that they are saddle-foci of type ($+/--$) and ($-/++$), respectively, for the former and latter steady states. Moreover, the three-element circuit is hampered and one can visualize three negative two-element feedback circuits formed by the products of matrix elements ($a_{12} = 1 - 3Y^2 + c, a_{21} = -c$), ($a_{31} = -c, a_{13} = 1 - 3X^2 + c$) and ($a_{23} = 1 - 3Z^2 + c, a_{32} = -c$) dominates at high values of c and control the dynamics. These circuits form an unstable periodicity in the XY, XZ, and YZ planes, respectively. The resultant of these periodic planar motions is also an unstable periodic motion in a plane perpendicular to the direction of the applied magnetic field. It hence explains the origin of the quasi-periodic attractor as before in the modified system in an isotropic external magnetic field.

4. Dynamics under dissipation (b ≠ 0)

This section considers the effect of frictional damping on a modified Thomas oscillator introduced via a constant damping function b. The idea of damping in the present context becomes prominent if we take an analogy of a charged particle moving a 3D lattice under the action of an energy source. Then the small value of b means that the spacing between the lattice points is significantly large. As a result, the chance of inelastic collision with the lattice points due to relative motion will be less frequent for a charged particle. This will give freedom to the particle to move around. On the other hand, for a high value of b, the 3D lattice will be densely packed, and as a consequence, the inelastic collision of the charged particle with the lattice will be frequent. As a result, it will lose energy and ultimately come to rest, making the motion impossible. This damping term is proportional to the velocity. A non-zero value of b makes the system non-conservative.

In earlier work, Lord Rayleigh[14] had shown that the velocity-dependent damping function leads to non-linear oscillations. This Velocity-dependent damping function has two domains: negative friction and positive friction. The latter kind of damping corresponds to energy dissipation, while the former corresponds to energy supply. The negative friction generates limit-cycle oscillation for various natural oscillators, symbolising self-sustained non-dissipative dynamics by balancing energy gain and dissipation. In addition, closed trajectories in non-linear non-conservative systems are attractive because they imply periodic motion.

However, Rayleigh-type oscillators were free oscillators as opposed to the present case, where there is a constant energy source. With the damping parameter switched ON, we have one more control parameter to study the dynamics and bifurcation of the modified system. To understand the interplay between damping, the oscillatory nature of the system, feedback circuits and the constant external Lorentz force, we consider only the weak damping case with b very small($b = 0.05$) for there is significant movement of the particle.

4.1. Unidirectional magnetic field

For this study, first of all, we consider the unidirectional applied magnetic field as modelled by equation (8) for a positive charge. We use a low value (0.05) for which the bifurcation diagrams and Lyapunov spectra are given below in figure 11.

For weak damping ($b = 0.05$), the oscillator dynamics is such that for higher values of the field strength parameter, there is a limit cycle in the range $c > 0.72$. The Largest Lyapunov exponent is zero throughout this range(blue). With $c = 0.8$, the system undergoes a period-doubling cascade and becomes chaotic for a further decrease in the field strength parameter($c < 0.72$). There are many quasi-periodic windows in the range $c = 0.13$ to 0.72 while a single quasi-periodic window between $c = 0.72$–0.8. In the former periodic windows, there exists a chaotic phenomenon with a finite life span known as transient chaos[15]. Finally, in the range $c = 0$ to $c = 0.13$ there is well-established chaos. Figure 12 shows the chaotic attractor for $b = 0.05$ for a fixed value of the field strength parameter $c = 0.53$.

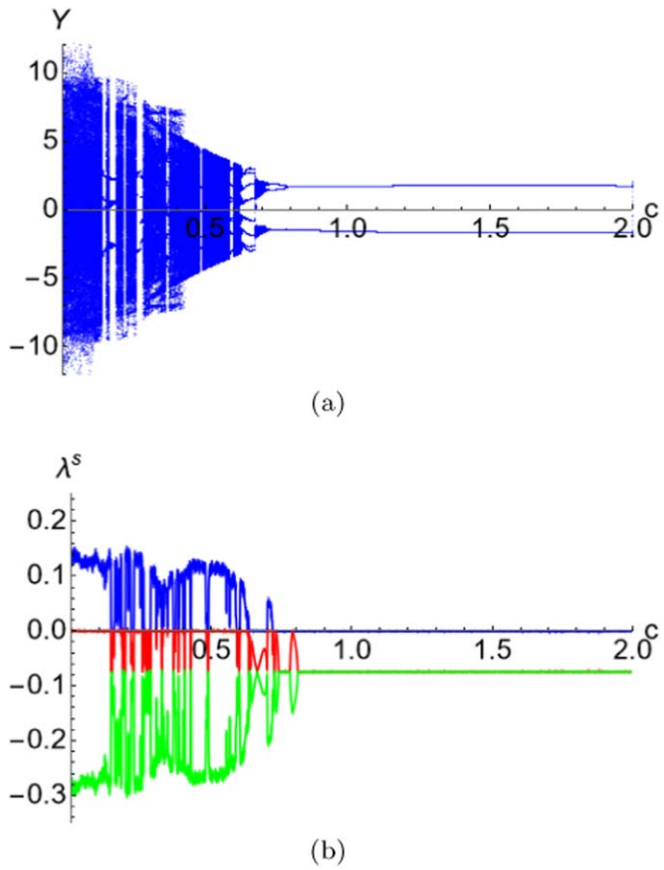

Figure 11. The figure shows the(a) bifurcation diagram and (b) Lyapunov spectrum for weak damping($b = 0.05$) of a charged Thomas oscillator with a unidirectional magnetic field.

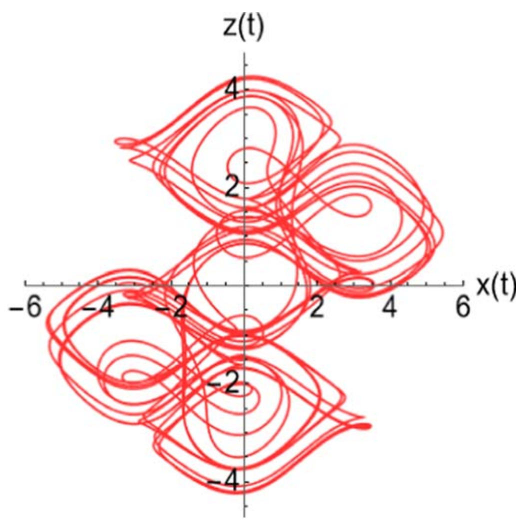

Figure 12. The figure shows the projection of chaotic attractor for $b = 0.05$ of a charged Thomas oscillator with unidirectional magnetic field strength $c = 0.53$.

4.2. Isotropic magnetic field

Now, look at equation (9) for the isotropic magnetic field applied to a charged Thomas system with a positive charge. For the weak damping value, we have the following scenarios: oscillations prevail in the system for higher values of parameter c with damping $b = 0.05$. The system jumps to a period-doubling cascade towards chaos for a c value around 0.35, where it undergoes a Hopf bifurcation. There are complex quasi-periodic windows confined to a small parameter range in the chaotic sea specified by $c < 0.32$. The largest Lyapunov exponent is

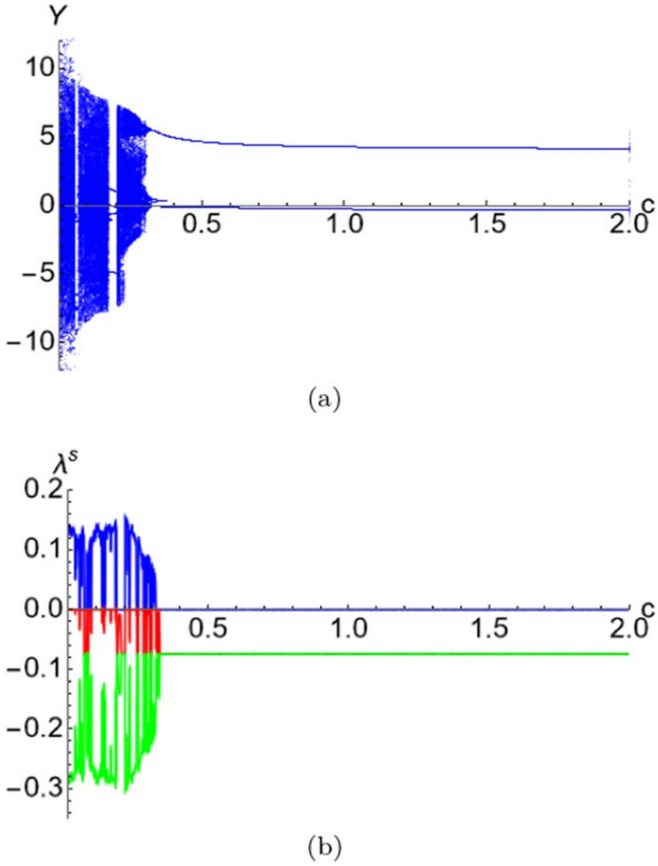

Figure 13. The figure shows the(a) bifurcation diagram and (b) Lyapunov spectrum for weak damping($b = 0.05$) of a charged Thomas oscillator with an isotropic magnetic field.

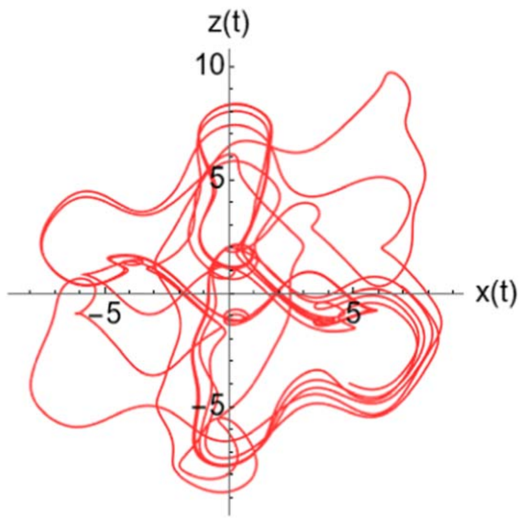

Figure 14. The figure shows the projections of the chaotic attractor for $b = 0.05$ of a charged Thomas Oscillator with isotropic magnetic field strength $c = 0.069$.

zero(blue) in the range $c > 0.32$ showing the existence of a limit cycle in this range and is confirmed by the bifurcation diagram as shown in figure 13. The chaotic attractor for $b = 0.05$ and a fixed value of the field strength parameter $c = 0.069$ is shown in figure 14.

The autocorrelation(ACF) function for the weak damping and weak field strength is studied and is shown in figure 15 for the unidirectional magnetic field. It is defined as in equation (19)

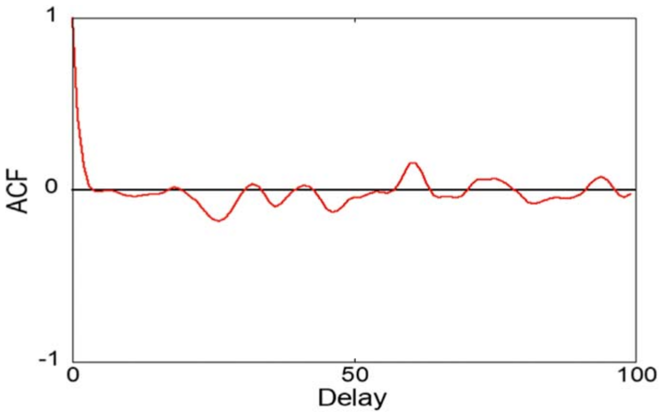

Figure 15. The figure shows the autocorrelation function for $b = 0.001$ of a charged Thomas Oscillator with unidirectional magnetic field strength $c = 0.001$.

$$ACF = \frac{\int_{\mathcal{T}}^{\infty} \mathcal{F}(t)\mathcal{F}(t - \mathcal{T})dt}{\int_{\mathcal{T}}^{\infty} \mathcal{F}(t)^2 dt} \tag{19}$$

where, $\mathcal{F}(t)$ is some dynamical variable, and $\mathcal{T}$ is the delay. Following [3] & [16], the autocorrelation function is calculated for $\dot{x}(t)$ rather than $x(t)$ itself since the mean of $\dot{x}$ is more nearly zero. Here we observe a short-time correlation and then a mild fluctuation around zero, as shown in figure 15. The correlation length in time will increase with the increase in control parameters. After the finite correlation time, the system shows random behaviour and can be connected to chaotic walks and diffusion phenomena. This random behaviour is prevalent in this deterministic system for weak damping and field strength. We found more or less the same result for the isotropic magnetic field for the choice of parameter values as specified in figure 15.

4.3. The effect of the feedback circuit & Lorentz forcing in the dissipative case

To explore the role of frictional damping, feedback, and external forcing for the unidirectional magnetic field equation (11) is again replaced by its caricature model without altering the qualitative nature of the dynamics. The Jacobian matrix is given by

$$\mathbf{Df_{uni}} \approx \begin{bmatrix} -b & 1 - 3Y^2 + c & 0 \\ -c & -b & 1 - 3Z^2 \\ 1 - 3X^2 & 0 & -b \end{bmatrix} \tag{20}$$

In this matrix, all the diagonal entries are negative damping constants. At $c = 0$, and for $b = 1$ there is only one stable fixed point. When the value of b decreases below 1, twenty-seven saddle foci of type $(+/--)$ and $(-/++)$ pop up in phase space which is verified from the eigenvalues structures at these steady states. The trajectories can percolate these steady states depending on the nature of the three-element feedback. One can explore the complex dynamics by decreasing the value of b. In the case, $b = 0.05$ and smoothly varying c, one can study the complex dynamics of the charged Thomas oscillator.

If we look at the dynamics of the modified system for fixed b$(= 0.05)$ and $c = 0$, there are fifteen fixed points of the type $(+/--)$ and $(-/++)$, and these are found to be unstable saddle foci from the eigenvalue structure. The trajectories percolate these fixed points depending on the feedback circuit's nature, and we have chaotic motion. With a smooth increase in the values of c from zero, the size of the attractor in phase space decreases along with the number of fixed points. But the chaotic nature persists up to $c \approx 0.72$, where the three-element single feedback circuit is the reason for chaoticity. It is also observed that there are only three fixed points for a wide range of c. For $c > 0.72$, we have a limit cycle around an unstable spiral point confirmed by the eigenvalues ($\lambda_1 = -\alpha_1, \lambda_2 = \alpha_2 + i\beta_2, \lambda_3 = \alpha_2 - i\beta_2$). If the eigenvalues are calculated in the vicinity of this steady state, the real part of the complex eigenvalues will be negative. This indicates that the stable periodic oscillation for $c > 0.72$ emerges only for the non-linear effects in the system. This, in turn, means that for these parameter values, the two-element feedback circuit dominates, and its stability is ensured by the negative diagonal terms a_{11} and a_{22} of the Jacobian matrix in (20). The negative diagonal term a_{33} shows that the limit cycle is attractive along the z-direction. This will result in a slight tilt in the orientation of the limit cycle about the z direction such that it is not ideally parallel to the XY plane figure 16.

The particle will lose energy due to in-elastic collision with the lattice points for the two cases mentioned above(unidirectional and isotropic). As a result, its kinetic energy will be converted into heat energy. The change in kinetic energy causes a rate of decrease in particle velocity, causing energy dissipation as electromagnetic

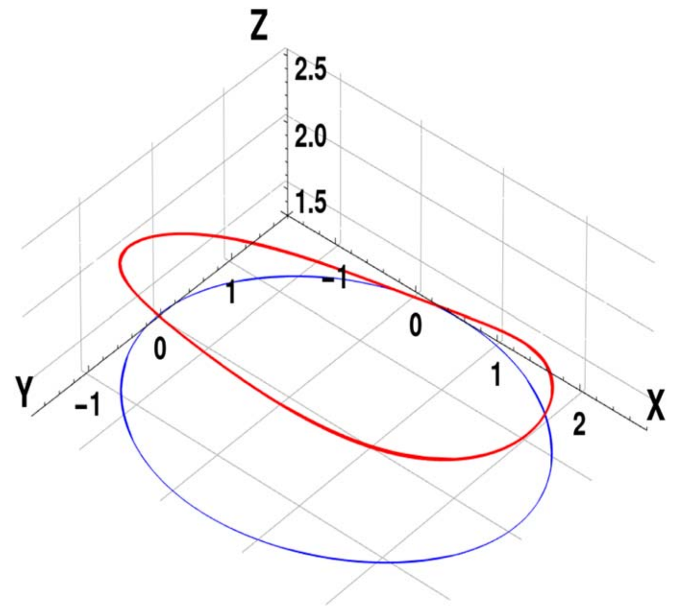

Figure 16. The limit cycle in phase space(red) and its projection to *xy*-plane(blue) for the dissipative case with unidirectional applied magnetic field ($b = 0.05, c = 1.5$).

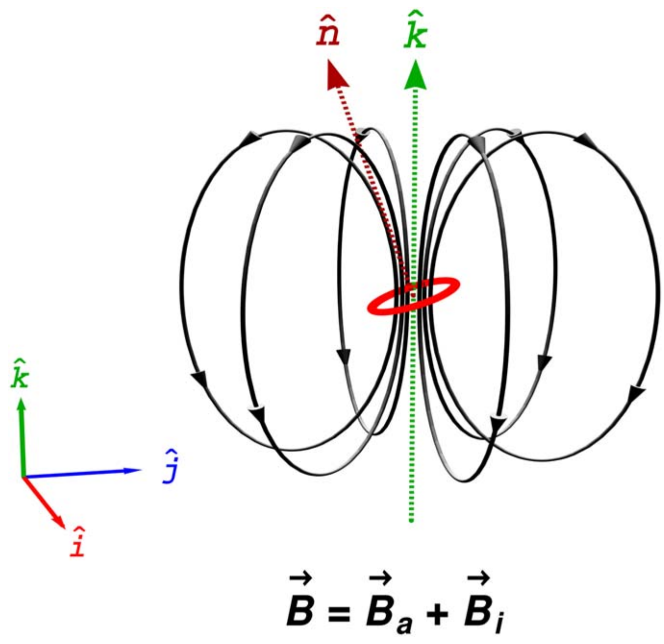

Figure 17. Artistic Representation of the limit-cycle with Magnetic Field Lines under dissipation ($b \neq 0$). $\hat{k}$ is a unit vector in the direction of the applied magnetic field, and $\hat{n}$ is the normal unit vector to the area enclosed by the loop.

radiation. The change in the particle's velocity means that the particle's trajectory is not a perfect circle. Instead, it is found to be a tilted path in the phase space. In other words, the plane of the limit cycle is not precisely parallel to the *XY plane*. As a result, magnetic flux changes for a charged particle moving on this limit cycle making the effective background magnetic field unsteady or time-varying as depicted in figure 17. This causes an energy exchange with the applied field. When there is a slight decrease in the magnetic flux, the particle dissipates; when there is an increase in magnetic flux, it absorbs energy from the environment. However, in both cases, the prominent dynamical feature is an emergent phenomenon in phase space, i.e. limit-cycle oscillations at high values of c. So one can infer the particle is absorbing energy from the constant energy source to perform self-sustained oscillations.

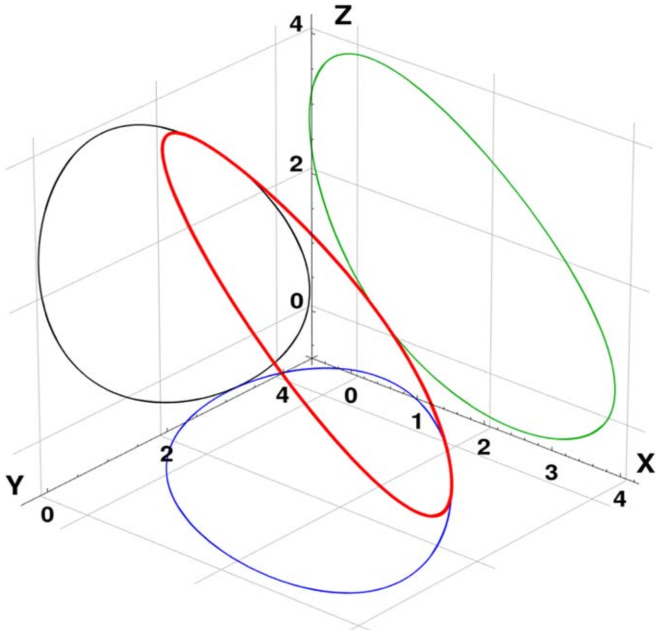

Figure 18. The limit cycle in phase space(red) and its projections to *xy*-plane(blue), *yz*-plane(black) and *xz*-plane(green) for the dissipative case with isotropic applied magnetic field ($b = 0.05$, $c = 1.5$).

A similar procedure can be followed in the isotropic case with damping to figure out the dynamics and its transition to a stable limit cycle. The origin of chaotic dynamics for the range $c < 0.32$ and the limit cycle for $c > 0.32$ can be related to feedback circuits. Notably, the limit cycle corresponds to a stable periodic oscillation due to non-linear effects in the caricature of equation (12). The limit cycle is located around an unstable spiral point. This periodicity results from the combination of these periodicities in the *XY*, *YZ*, and *XZ* planes, as in figure 18.

5. Results and discussions

The additional property, charge, given to the Thomas oscillator in the presence of a DC magnetic field converts the system from a chaotic oscillator with feedback to a self-excited oscillator[17]. Now the particle can absorb and dissipate energy in its environment and can be called an active particle. Self-excited systems are characterized by an external energy supply, an oscillating system, and a feedback and control system that regulate the energy flow in and out of the system. This is indeed the case here for the modified Thomas oscillator.

In the case of conserved dynamics of the charged Thomas oscillator, the transport of charge from one point to another is such that it is trying to preserve the phase-space volume locally. The local curving of the trajectories in the phase space is such that the stretching and folding of the trajectories keep the phase volume conserved. Secondly, the motion of the charge is such that in the presence of an external magnetic field, the kinetic energy of the charge is converted into internal magnetic energy. Then this energy is liberated in the form of particle acceleration and radiation. The third important point is that an external magnetic field tends to orient a current loop such that the field of the circuit points in the direction of the applied magnetic field. Finally, the topology of the quasi-periodic attractor can be changed by choice of the initial condition, from a straight line to a torus for the isotropic external magnetic field. The properties mentioned above are essential in understanding the cause of events involved in plasma physics and astrophysics. Cosmic magnetic structures such as terrestrial magnetic storms, solar flares and radio pulsar wind are a few examples where the dynamics of charged Thomas oscillators can play a paramount role in understanding.

In this study, for the conservative dynamics of the charged Thomas particle in an external magnetic field, we find that the particle is trapped with quasi-periodic oscillations at high field strength parameter values. The external magnetic field confines a charged particle within a bounded space domain. In this situation, slowly varying dynamical quantities exist, aka adiabatic invariants, which leads the system dynamics very close to the regular motion. When the adiabatic invariants are destroyed, the dynamics become chaotic. So one can say that there is particle confinement through a magnetic adiabatic trap. The adiabatic trap can prevent particle loss along the magnetic field direction. Such magnetic traps are cornerstones of modern ultra-cold physics, reactor physics, quantum information processing, quantum optics, and quantum metrology.

The modified Thomas system in the dissipative regime is an excellent candidate for understanding Brownian motion's microscopic origin. As we saw, the particle interacts non-linearly with its surroundings and generates local instability. This local instability is the reason for randomness whose origin is perfectly deterministic. This leads to a diffusion process whose origin is chaos. The external drive suppresses the randomness in the system at high field strength values bringing order to the system. From a practical point of view, the properties of the system are such that it will find application in biophysics and the modelling of novel materials. They are effortless to detect and navigate since they emit radiation, and their trajectories can be controlled by applying an external DC magnetic field. These particles can consume energy from the surroundings and perform useful work, like directed motion. In all these cases, control is achieved by regulating the feedback mechanism. So the modified Thomas system will be a good candidate for understanding the dynamics of such micro-particles.

6. Conclusion

This paper considers a charged Thomas oscillator, assuming it is placed in a constant external magnetic field. The system is modelled by modifying the Thomas oscillator, and dynamics are studied by considering two regimes of application; the conservative and dissipative for weak and strong magnetic fields. In both cases, when the system is in a chaotic regime, the applied magnetic field shifts the dynamics to a neighbouring level ($chaos \longrightarrow quasi - periodicity \longrightarrow limit\text{-}cycle$). A non-invasive control over the complex behaviour of the oscillator is achieved and could bring in a balance between energy gain and energy loss. The role played by self-excitation in controlling frictional damping is unveiled in terms of feedback circuits and steady states. In all cases, the interplay between dynamics, the external field, and the induced field leads to exciting properties. In the conservative regime, the route to chaos is via non-linear resonance, and an adiabatic particle trapper is realized. The transition from adiabatic motion to chaos is through the destruction of adiabatic invariance. In the dissipative regime, chaos can be controlled by applying the external field. The transition to chaos, in this case, is via period-doubling cascades. In both cases, introducing the forcing term has a new negative feedback circuit that forces the system dynamics to stable periodic or quasi-periodic oscillations for a broad range of parameter values. Finally, the model highlights the important role of non-linear resonance and chaos in the microscopic origin of Brownian motion. In the conservative case, non-linear resonance leads to non-integrability and chaos for decreasing field strengths. In the dissipative case, the system shows sensitive dependence to initial conditions for low damping and decreasing field strength. In both these cases, the particle interacts with the field and this communication can be controlled by regulating the field strength parameter. The dynamics of the collection of such particles with suitable coupling functions may lead to spatio-temporal pattern formation.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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